**CO - 1**

**Session 1 - Introduction to Mathematical Programming**

The Mathematical Programming is part of operations research which is the systematic application of quantitative methods, techniques and tools to the analysis of problems involving the operation of systems.

Operations Research is the science of rational decision-making and the study, design and integration of complex situations and systems with the goal of predicting system behavior and improving or optimizing system performance. The British/Europeans refer to "operational research", the Americans to "operations research" -but both are often shortened to just "OR" which is the term we will use.

In the decades after the world war two, the tools of operations research were more widely applied to problems in business, industry and society. Since that time, operational research has expanded into a field widely used in industries ranging from petrochemicals to airlines, finance, logistics, and government, moving to a focus on the development of mathematical models that can be used to analysis and optimize complex systems, and has become an area of active academic and industrial research.

**Linear Programming Problem (LP Problem)**

Linear Programming is a mathematical optimization technique, by optimization technique it refers to a method which attempts to maximize or minimize some objective, for example, maximize profits or minimize costs. Linear programming is a subset of a larger area of mathematical optimization procedures called mathematical programming, which is concerned with making an optimal set of decisions.

In any LP problem, certain decisions need to be made.  These decisions are represented by decision variables which are used in the formulation of the LP model.

**Basic Structure of a Linear Programming Problem**

The basic structure of an LP problem is either to maximize or minimize an objective function, while satisfying a set of constraining conditions called constraints.

**Objective function:**  The objective function is a mathematical representation of the overall goal stated in terms of the decision variables. The firm’s objective and its limitations must be expressed as mathematical equations or inequalities, and these must be linear equations and inequalities.

**Constraints:**  The constraints are also stated in terms of the decision variables and represent conditions which must be satisfied in determining the values of the decision variables. Most constraints in a linear programming problem are expressed as inequalities. They set upper or lower limits, they do not express exact equalities thus, permit many possibilities.

Resources must be in limited supply. For example, a furniture plant has a limited number of machine-hours available; consequently, the more hours it schedules for furniture’s, the fewer furniture’s it can make. There must be alternative courses of action, one of which will achieve the objective.

**Assumptions Linear Programming**

When using Linear Programming to solve a real engineering and business problem, five assumptions have to be made.

**Linearity:**  The objective function and constraints are all linear functions that is every term must be of the first degree.  Linearity implies the next two assumptions.

**Proportionality:**  For the entire range of the feasible output, the rate of substitution between the variables is constant.

**Additivity:** All operations of the problem must be additive with respect to resource usage, returns, and cost.  This implies independence among the variables.

**Divisibility:** Non-integer solutions are permissible.

**Certainty:**  All coefficients of the LP model are assumed to be known with certainty.

**Advantages and limitations of LPP:**

LP has been considered an important tool due to following reasons:

1. LP makes logical thinking and provides better insight into business problems.
2. Manager can select the best solution with the help of LP by evaluating the cost and profit of various alternatives.
3. LP provides an information base for optimum alloca­tion of scarce resources.
4. LP assists in making adjustments according to changing conditions.
5. LP helps in solving multi-dimensional problems.

LP approach suffers from the following limitations also:

1. This technique could not solve the problems in which variables cannot be stated quantitatively.
2. In some cases, the results of LP give a confusing and misleading picture. For example, the result of this technique is for the purchase of 1.6 machines. It is very difficult to decide whether to purchase one or two- machine because machine can be purchased in whole.
3. LP technique cannot solve the business problems of non-linear nature.
4. The factor of uncertainty is not considered in this technique.
5. This technique is highly mathematical and complicated.
6. If the numbers of variables or constraints involved in LP problems are quite large, then using costly electronic computers become essential, which can be operated, only by trained personnel.
7. Under this technique to explain clearly the objective function is difficult.

The basic problem before any manager is to decide the manner in which limited resources can be used for profit maximization and cost minimization. This needs best allocation of limited resources, for this purpose linear programming can be used advantageously.

**Session 2 - Mathematical Modeling of Linear Programming Problem**

The procedure for mathematical formulation of a linear programming problem consists of the following major steps:

Step:1: Study the given situation to find the key decisions to be made.

Step:2: Identify the variables involved and designate them by symbols xj (j=1, 2, …).

Step:3: State the feasible alternatives which generally are xj 0, for all j.

Step 4: Identify the constraints in the problem and express them as linear inequalities or equations, LHS of which are linear functions of the decision variables.

Step 5: Identify the objective function and express it as a linear function of the decision variables.

**Example1:**   A transport company has two types of trucks, Type A and Type B. Type A has refrigerated capacity of 20 m3 and a non-refrigerated capacity of 40 m3 while Type B has refrigerated capacity of 30 m3 and non-refrigerated capacity of 30 m3. A grocer needs to hire trucks for the transport of 3,000 m3 of refrigerated stock and 4 000 m3 of non-refrigerated stock. The cost per kilometer of a Type A is $30 and $40 for Type B. How many trucks of each type should the grocer rent to achieve the minimum total cost (formulating Mathematical modelling of LPP).

**Solution:** x = no. of Type A trucks, y = no. of Type B trucks

Min Z = 30x+40y Subjected to the constraints: 20x+30y ≥ 3000, 40x+30y ≥ 4000, x, y ≥ 0

**Example2:**   A store wants to liquidate 200 of its shirts and 100 pairs of pants from last season. They have decided to put together two offers, A and B. Offer A is a package of one shirt and a pair of pants which will sell for $30. Offer B is a package of three shirts and a pair of pants, which will sell for $50. The store does not want to sell less than 20 packages of Offer A and less than 10 of Offer B. How many packages of each do they have to sell to maximize the money generated from the promotion (formulating Mathematical modelling of LPP).

**Solution:** x = number of packages of offer A, y = number of packages of offer B

Max Z= 30x+50y Subjected to the constraints: x+3y ≤ 200, x + y ≤ 100, x ≥ 20, y ≥ 10, x, y ≥ 0

**Problems to be discussed by the faculty:**

1. The owner of a shop producing automobile trailers wishes to determine the best mix for

his three products: flat-bed trailers, economy trailers, and luxury trailers. His shop is limited to working 24 days/month on metalworking and 60 days/month on woodworking for these products. The following table indicates production data for the trailers:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *Usage per unit of trailer Resources* | | | *Resources*  *availabilities* |
|  | *Flat - bed* | *Economy* | *Luxury* |
| Metalworking days | 1/2 | 2 | 1 | 24 |
| Woodworking days | 1 | 2 | 4 | 60 |
| Contribution ($ × 100) | 6 | 14 | 13 |  |

Formulate the mathematical modelling of the linear programming problem.

2. Suppose that a custom molder has one injection-molding machine and two different dies to fit the machine. Due to differences in number of cavities and cycle times, with the first die he can produce 100 cases of six-ounce juice glasses in six hours, while with the second die he can produce 100 cases of ten-ounce fancy cocktail glasses in five hours. He prefers to operate only on a schedule of 60 hours of production per week. He stores the week’s production in his own stockroom where he has an effective capacity of 15,000 cubic feet. A case of six-ounce juice glasses requires 10 cubic feet of storage space, while a case of ten-ounce cocktail glasses requires 20 cubic feet due to special packaging. The contribution of the six-ounce juice glasses is $5.00 per case; however, the only customer available will not accept more than 800 cases per week. The contribution of the ten-ounce cocktail glasses is $4.50 per case and there is no limit on the amount that can be sold. How many cases of each type of glass should be produced each week in order to maximize the total contribution (Formulate the Mathematical modelling of LPP).

**Practice problems**

1. A firm manufactures 2 types of products A & B and sells them at a profit of $2 on type A & $3 on type B. Each product is processed on 2 machines G & H. Type a requires 1 minute of processing time on G and 2 minutes on H. Type B requires 1 minute on G & 1 minute on H. The machine G is available for not more than 6 hrs. 40 mins., while machine H is available for 10 hrs. during any working day. Formulate the problem as LPP.
2. A transport company has two types of trucks, Type A and Type B. Type A has refrigerated capacity of 20 m3 and a non-refrigerated capacity of 40 m3 while Type B has refrigerated capacity of 30 m3 and non-refrigerated capacity of 30 m3. A grocer needs to hire trucks for the transport of 3,000 m3 of refrigerated stock and 4 000 m3 of non-refrigerated stock. The cost per kilometer of a Type A is $30 and $40 for Type B. How many trucks of each type should the grocer rent to achieve the minimum total cost (formulating Mathematical modelling of LPP).
3. A dietician wishes to mix two types of foods in such a way that vitamin contents of the mixture contain atleast 8 units of vitamin A and 10 units of vitamin C. Food ‘I’ contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C. Food ‘II’ contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs Rs 50 per kg to purchase Food ‘I’ and Rs 70 per kg to purchase Food ‘II’. Formulate this problem as a linear programming problem to minimise the cost of such a mixture.
4. A company produces 2 types of cowboy hats. Each hat of the first type requires twice as much labour time as the second type. The company can produce a total of 500 hats a day. The market limits the daily sales of first and second types to 150 and 250 hats. Assuming that the profits per hat are $8 per type A and $5 per type B, formulate the problem as Linear Programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.
5. A cooperative society of farmers has 50 hectares of land to grow two crops X and Y. The profit from crops X and Y per hectare are estimated as Rs 10,500 and Rs 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops X and Y at rates of 20 litres and 10 litres per hectare. Further, no more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so as to maximise the total profit of the society? (formulating Mathematical modelling of LPP).

**Session 3- The Geometry of Linear Optimization**

The linear programming problems of two decision variables can be easily solved by graphical method.

The outlines of the graphical procedure as follows:

**Step: 1** Identify the problem-the decision variables, the objective and the restrictions.

**Step: 2** set up the mathematical formulation of the problem.

**Step: 3** consider each inequality –constraint as an equation.

**Step: 4** Plot each equation on the graph, as each one will geometrically represent a straight line.

**Step: 5** Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality constraint corresponding to that line is ‘≤’, then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality-constraint with ‘≥’ sign, the region above the line in the first quadrant is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region thus obtained is called the **feasible region**.

**Step: 6**: Choose the convenient value of z(say-0) and plot the objective function line.

**Step: 7** pull the objective function line until the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

**Step: 8** Read the coordinates of the extreme point(s) selected in step6, and find the maximum or minimum value of z.

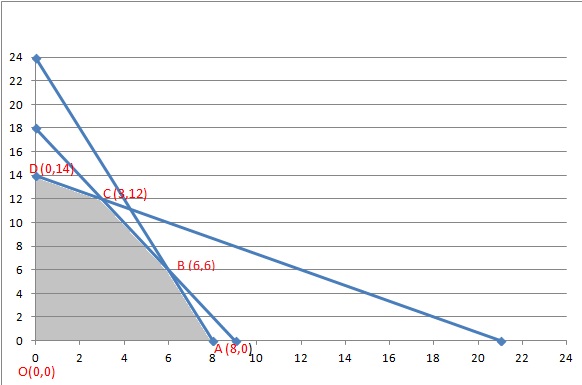
**Example 1:**

Using the graphical method to solve the following problem:

Subject to constrains, ,

Solution:

First, we draw the lines of all the given equations and shade the common region according to the signs.



The feasible region is the intersection of the regions defined by the set of constraints and the coordinate axis (conditions of non-negativity of variables). This feasible region is represented by the O-A-B-C-D-O

As a feasible region exists, extreme values (or polygon vertices) are calculated. These vertices are the point’s candidate as optimal solutions. In the example, these points are O, A, B, C, and D, as shown in the figure.

Finally, the objective function is evaluated in each of these points (results are shown in the tableau below). Since C-point provides the greatest value to the Z-function and the objective is to maximize, this point is the optimal solution: Z = 33 with x = 3 and y = 12.

|  |  |  |
| --- | --- | --- |
| Points | Coordinates | Value of Objective function |
| O | (0,0) | 0 |
| A | (8,0) | 24 |
| B | (6,6) | 30 |
| C | (3,12) | 33 |
| D | (0,14) | 28 |

**Problems to be discussed by the faculty:**

1. Indicate graphically whether each of the following linear programs has a feasible solution. Graphically determine the optimal solution, if one exists, or show that none exists.

a) Maximize *z* = *x*1 + 2*x*2, subject to: *x*1 − 2*x*2 ≤ 3, *x*1 + *x*2 ≤3, *x*1, *x*2 ≥ 0.

b) Minimize *z* = *x*1 + *x*2, subject to: *x*1 -*x*2 ≤ 2, *x*1 -*x*2 ≥ −2, *x*1, *x*2 ≥ 0.

c) Maximize *z* = *x*1 + *x*2, subject to: *x*1 -*x*2 ≤ 2, *x*1 -*x*2 ≥ −2, *x*1, *x*2 ≥ 0.

d) Maximize *z* = 3*x*1 + 4*x*2, subject to: *x*1 − 2*x*2 ≥ 4, *x*1 + *x*2 ≤ 3, *x*1, *x*2 ≥ 0.

**Practice problems:**

1. Consider the following linear program: Maximize *z* = 2*x*1 + *x*2 subject to: 12*x*1 + 3*x*2 ≤ 6, −3*x*1 + *x*2 ≤ 7, *x*2 ≤ 10, *x*1, *x*2 ≥ 0. Draw a graph of the constraints and shade in the feasible region. Label the vertices of this region with their coordinates.

2. Consider the following linear program and use the graphical method to solve: Maximize *z* = 3*x*1+ 2*x*2, subject to –*x*1+ 2*x*2 ≤ 30, 2*x*1+ *x*2 ≤ 40, *x*1+ 2*x*2 ≥ 10, *x*1 ≥ 0, *x*2 ≥ 0

3. A manufacturer produces two types of models M1 and M2. Each M1 model requires 4 hours of grinding and 2 hours of polishing, whereas each M2 model requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works for 40 hours a week and each polisher works for 60 hours a week. Profit on M1 model is Rs. 3.00 and on an M2 model is Rs. 4.00. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week? Formulate the given problem as a LPP and also obtain the graphical solution.

4. The manufacture of patent medicines is proposed to prepare a production plan for medicines A and B. There is sufficient ingredient available to make 20,000 bottle or medicine A and 40,000 bottles of medicine B, but there are only 45,000 bottles into which either of the medicines can be filled. Further, it takes three hours to prepare enough material to fill 1000 bottles of medicine A and one hour to prepare enough material to fill 1000 bottles of medicine B, and there are 66 hours available for this operation. The profit is Rs 8/- per bottle for medicine A and Rs.7/- per bottle for medicine B. Formulate this problem as a

a) Linear programming problem

b) How the manufacturer schedules his production in order to maximize the profit using graphical method.

**Session 4 - Special cases that arise the application of the graphical method**

1. **Alternative optima:** When the objective function is parallel to a binding constraint (i.e., a constraint that is satisfied as an equation by the optimal solution), the objective function will assume the same optimum value at more than one solution point. For this reason, they are called alternative optimal.
2. **Unbounded solution:** When the values of the decision variables may be increased indefinitely without violating any of the constraints, the solution space (feasible region) is unbounded. The value of objective function, in such cases, may increase (for maximization) or decrease (for minimization) indefinitely. Thus, both the solution space and the objective function value are unbounded.
3. **Infeasible solution (or non-existing) solution:**  When the constraints are not satisfied simultaneously, the linear programming problem has no feasible solution. This situation can never occur if all the constraints are of the ≤ type.

**Problems to be discussed by the faculty:**

1. Use graphical method to solve the LPP: Maximize Z= 2x1+4x2 Subject to the constraints: x1+2x2≤5, x1+x2≤4; x1, x2≥0.

2. Use graphical method to find the maximum value of z=x1+2x2 Subject to the conditions: x1-x2≤1, x1+x2≥3; x1≥0, x2≥0.

3. Use graphical method to find the maximum value of z=3x1+2x2Subject to the conditions:

2x1+x2≤2, 3x1+4x2≥12; x1≥0, x2≥0.

**Session 5 & 6 - Simplex Method**

The Simplex method is an iterative (i.e., step-by step) procedure by which a new basic feasible solution can be obtained from a given (initial) basic feasible solution so that the value of the objective function is improved.

**Simplex Algorithm:**

**Step:1** Check whether the objective function of the given L.P.P is to be maximized or minimized. If it is to be minimized then we convert it into a problem of maximizing it by using the result Minimum Z = - Maximum(-z).

**Step: 2** Check whether all bi (i=1, 2, ..., m) are non-negative. If any one of bi is negative then multiply the corresponding in equation ofthe constraints by -1, so as to get all bi (i=1, 2, ..., m) non-negative.

**Step: 3** convert all the inequations of the constraints into equation by introducing slack/surplus and artificial variables in the constraints. Put the costs of slack/surplus variables equal to zero and the cost of artificial variable -1 or M (depending on the method) in modified objective function.

**Step: 4** Obtain an initial basic feasible solution by setting x1= x2 =...= xn = 0 in equations obtained in step 3.

**Step: 5** Determine which variable to enter into the solution next. Identify the column—

hence the variable—with the largest positive number in the *Cj* \_ *Zj* row of the previous

tableau. This step means that we will now be producing some of the product contributing

the greatest additional profit per unit.

**Step: 6** Determine which variable to replace. Because we have just chosen a new variable to enter into

the solution, we must decide which variable currently in the solution to remove to make

room for it. To do so, we divide each amount in the quantity column by the corresponding

number in the column selected in step 5. The row with the *smallest nonnegative number* calculated in this fashion will be replaced in the next tableau (this smallest number, by the way, gives the maximum number of units of the variable that we may place in the solution). This row is often referred to as the **pivot row**, and the column identified in step 5 is called the **pivot column**. The number at the intersection of the pivot row and pivot column is the **pivot number**.

**Step 7:** Compute new values for the pivot row. To find them, we simply divide every number in the row by the *pivot number*.

The new coefficients of the tableau are calculated as follows:

1. In the pivot row each new value is calculated as: *New value = Previous value / Pivot*
2. In the other rows each new value is calculated as:

*New value = Previous value - (Previous value in pivot column \* New value in pivot row)*

**Step 8:** Compute the *Zj* and *Cj* \_ *Zj* rows, as demonstrated in the initial tableau. If all numbers in

the *Cj* \_ *Zj* row are zero or negative, we have found an optimal solution. If this is not the

case, we must return to step 5.

**Example 1:**

Solve linear programming problem Max z =500 x1 + 450 x2, subjected to the constrains 6x1 + 5x2 ≤ 60, x1 + 2x2 ≤ 15; x1 ≤ 8, x1, x2 ≥ 0.

Solution: Formulation of LPP

Max z =500 x1 + 450 x2 + 0 s1 + 0 s2 + 0 s3

6x1 + 5x2 + s1 + 0 s2 + 0 s3 = 60,

x1 + 2x2 + 0 s1 + s2 + 0 s3 = 15;

x1 + 0 x2 + 0 s1 + 0 s2 + 1 s3= 8,

x1, x2, s1, s2, s3 ≥ 0.

Table -1

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Basic Variables (BV) | Cost of BV | Cj | 500 | 450 | 0 | 0 | 0 | Min. Ratio |
| Value of BV (Xb) | x1 | x2 | s1 | s2 | s3 |
| s1 | 0 | 60 | 6 | 5 | 1 | 0 | 0 | 10 |
| s2 | 0 | 15 | 1 | 2 | 0 | 1 | 0 | 15 |
| s3 | 0 | 8 | 1 | 0 | 0 | 0 | 1 | 8 |
| Z= 0 | | Zj | 0 | 0 | 0 | 0 | 0 |  |
| Cj - Zj | 500 | 450 | 0 | 0 | 0 |

Since Max {Cj -Zj} = 500 therefore the corresponding column x1 enter into the basis and from the minimum ratio column, the Min {min. ratio} = 8 therefore the corresponding row s3 leave from the basis.

Table -2

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Basic Variables (BV) | Cost of BV | Cj | 500 | 450 | 0 | 0 | 0 | Min. Ratio |
| Value of BV (Xb) | x1 | x2 | s1 | s2 | s3 |
| s1 | 0 | **12** | **0** | **5** | **1** | **0** | **-6** | **12/5** |
| s2 | 0 | **7** | **0** | **2** | **0** | **1** | **-1** | **7/2** |
| x1 | 500 | **8** | **1** | **0** | **0** | **0** | **1** | **--** |
| Z = 4000 | | **Zj** | **500** | **0** | **0** | **0** | **500** |  |
| Cj - Zj | 0 | 450 | 0 | 0 | -500 |

Since Max {Cj -Zj} = 450 therefore the corresponding column x2 enter into the basis and from the minimum ratio column, the Min {min. ratio} = 12/5 therefore the corresponding row s1leave from the basis.

Table -3

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Basic Variables (BV) | Cost of BV | Cj | 500 | 450 | 0 | 0 | 0 | Min. Ratio |
| Value of BV (Xb) | x1 | x2 | s1 | s2 | s3 |
| x2 | 450 | **12/5** | **0** | **1** | **1/5** | **0** | **-6/5** | -- |
| s2 | 0 | **11/5** | **0** | **0** | **-2/5** | **1** | **7/5** | 11/7 |
| x1 | 500 | **8** | **1** | **0** | **0** | **0** | **1** | 8 |
| Z = 5080 | | **Zj** | **500** | **450** | **90** | **0** | **-40** |  |
| **Cj - Zj** | **0** | **0** | **-90** | **0** | **40** |

Since Max {Cj -Zj} = 40 therefore the corresponding column s3 enter into the basis and from the minimum ratio column, the Min {min. ratio} = 11/7 therefore the corresponding row s2 leave from the basis.

Table -4

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Basic Variables (BV) | Cost of BV | Cj | 500 | 450 | 0 | 0 | 0 | Min. Ratio |
| Value of BV (Xb) | x1 | x2 | s1 | s2 | s3 |
| x2 | 450 | **30/7** | **0** | **1** | **-1/7** | **6/7** | **0** |  |
| S3 | 0 | **11/7** | **0** | **0** | **-2/7** | **5/7** | **1** |  |
| x1 | 500 | **45/7** | **1** | **0** | **2/7** | **-5/7** | **0** |  |
| Z = 5142.86 | | **Zj** | **500** | **450** | **550/7** | **200/7** | **0** |  |
| **Cj - Zj** | **0** | **0** | **-550/7** | **-200/7** | **0** |

Since all Cj – Zj ≤ 0, therefore the iteration process end. The optimum solution LPP is x1 = 45/7, x2 = 30/7 and the Max z =5142.86.

**Comparison between maximisation case and minimisation case:**

|  |  |  |
| --- | --- | --- |
| S. no. | Maximisation case | Minimisation case |
| Similarities: | | |
| 1 | It has an objective function. | This too has an objective function. |
| 2 | It has structural constraints. | This too has structural constraints. |
| 3 | The relationship between variables and  constraints is linear. | Here too the relationship between and variables  constraints is linear. |
| 4 | It has non-negativity constraint. | This too has non-negativity constraints. |
| 5 | The coefficients of variables may be positive,   negative or zero. | The coefficient of variables may be positive,  Negative or zero. |
| 6 | For selecting out going variable (key row) lowest replacement ratio is selected. | For selecting out going variable (key row) lowest replacement ratio is selected. |

|  |  |  |
| --- | --- | --- |
| s. no. | Maximisation case | Minimisation case |
| Differences: | | |
| 1 | The objective function is of maximisation type. | The objective function is of minimisation type. |
| 2 | The inequalities are of ≤ type. | The inequalities are of ≥ type. |
| 3 | To convert inequalities into equations, slack variables are added. | To convert inequalities into equations, surplus Variables are subtracted and artificial surplus variables are added. |
| 4 | While selecting incoming variable, highest positive opportunity cost is selected from net evaluation Row. | While selecting, incoming variable, lowest element in the net evaluation row is selected (highest number with negative sign). |
| 5 | When the elements of net evaluation row is either Negative or zeros, the solution is optimal | When the element of net evaluation row is either positive or zeros the solution is optimal. |

**Problems to be discussed by the faculty:**

1. Solve the following problem using the simplex method:

Maximize *z* = −3*x*1 + 6*x*2,

subject to:

5*x*1 + 7*x*2 ≤ 35,

−*x*1 + 2*x*2 ≤ 2,

*x*1, *x*2 ≥ 0.

2. Solve the following problem using the simplex method:

Minimize *z* = *x*1 − 2*x*2 − 4*x*3 + 2*x*4,

subject to:

*x*1 − 2*x*3 ≤ 4,

*x*2 − *x*4 ≤ 8,

−2*x*1 + *x*2 + 8*x*3 + *x*4 ≤ 12,

*x*1, *x*2, *x*3, *x*4 ≥ 0.

**Practice problems:**

1. Two products are manufactured by passing sequentially through three machines. Time per machine allocated to the two products is limited to hours per day. The production time and profit per unit of each product are:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Product | Production time in (Minutes) | | | |
|  | M1 | M2 | M3 | Profit (Rs) |
| 1  2 | 12  4 | 10  8 | 5  10 | 20  30 |
| Availability | 150 | 300 | 120 |  |

Determine the Optimal mix of the two products using Simplex method.

2. A manufacture makes two products P1 and P2 using two machines M1 and M2. Product P1 requires 2 hours on machine M1 and 6 hours on machine M2. Product P2 requires 5 hours on machine M1 and no time on machine M2. There are 16 hours of time per day available on machine M1 and 30 hours on M2. Profit margin from P1 and P2 is Rs. 2 and Rs. 10 per unit respectively. Determine the optimum production mix to optimize profit.

3. Use simplex method to solve the following LPP:

Maximize 8*x*1 + 6*x*2,

subject to:

3*x*1 + 2*x*2 ≤ 28,

5*x*1 + 2*x*2 ≤ 42,

*x*1 ≤ 8,

*x*2 ≤ 8,

*x*1, *x*2 ≥ 0.

4. Use simplex method to solve the following LPP:

Maximize *Z* = 10*x* + 20*y*

subjected to the constrains:

5*x* + 3 *y* ≤ 30

3*x* + 6*y* ≤ 36

2*x* + 5*y* ≤ 20

Both *x* and *y* are ≥ 0.

5. Use simplex method to solve the following linear programming problem:

Maximize *z* = *x*1,

subject to:

−*x*1 + *x*2 ≤ 2,

*x*1 + *x*2 ≤ 8,

−*x*1 + *x*2 ≤ −4,

*x*1, *x*2 ≥ 0.

**Session 7 & 8 - Big-M Method**

The Big-M method is a method of solving a linear programming problem involving artificial variable. In this method we assign a very high penalty (say M) to the artificial variables in the objective function.

The iterative procedure of the algorithm is given below:

**Step: 1** Write the given LPP into its standard form and check whether there exists a starting basic feasible solution.

1. If there is ready starting basic feasible solution, move on to step 3.
2. If there does not exist a ready starting basic feasible solution, move on to step 2.

**Step: 2** Add artificial variable to the left side of each equation that has no obvious starting basic variables. Assign a very high penalty (say M) to these variables in the objective function.

**Step:3** Apply simplex method to the modified LPP following cases may arise at the last iteration:

1. At least one artificial variable is present in the basis with zero value. In such a case the current optimum basic feasible solution is degenerate.
2. At least one artificial variable is present in the basis with a positive value, in such a case, the given LPP does not possess an optimum basic feasible solution. The given problem is said to have a pseudo-optimum basic feasible solution.
3. No artificial variable in the basis and at least one 0, in such a case apply usual simplex algorithm to the modified simplex table to get the optimum solution of the original problem.

Example: Find solution using Big M method  
 Minimum Z = x1 + x2  
 subject to:  
 x1 + 2x2 ≥ 2  
 x1 + 7x2 ≥ 7   
 and x1, x2 ≥ 0

Solution:

Minimum Z = x1 + x2 + 0s2 + 0s2 + MA1 + MA2  
 subject to:  
 x1 + 2x2 – s2 +0s2 + 1A1 + 0 A2 = 2  
 x1 + 7x2 + 0s2 - s2 + 0A1 + 1 A2 = 7   
 and x1, x2, s1, s2, A1, A2 ≥ 0

**Table 1:**

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Basic  Variables (BV) | Cost of BV | Cj | 1 | 1 | 0 | 0 | M | M | Min. Ratio |
| Value of BV (Xb) | x1 | x2 | s1 | s2 | A1 | A2 |
| A1 | M | 2 | 1 | 2 | -1 | 0 | 1 | 0 | 1 |
| A2 | M | 7 | 1 | 7 | 0 | -1 | 0 | 1 | 1 |
| Z= 9M | | Zj | 2M | 9M | -M | -M | M | M |  |
| Cj - Zj | 1-2M | 1-9M | M | M | 0 | 0 |  |

Since Min {Cj -Zj} = 1-9M therefore the corresponding column x2 enter into the basis and from the minimum ratio column, the Min {min. ratio} = 1 (in the case of tie any one of the row can be selected) therefore the corresponding row A1 leave from the basis.

**Table 2:**

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Basic  Variables  (BV) | Cost of BV | Cj | 1 | 1 | 0 | 0 | M | M | Min. Ratio |
| Value of BV (Xb) | x1 | x2 | s1 | s2 | A1 | A2 |
| x2 | 1 | 1 | 1/2 | 1 | -1/2 | 0 | 1/2 | 0 | -- |
| A2 | M | 0 | -5/2 | 0 | 7/2 | -1 | -7/2 | 1 | 0 |
| Z=1 | | Zj | (1-5M)/2 | 1 | (7M-1)/2 | -M | (1-7M)/2 | M |  |
| Cj - Zj | (5M+1)/2 | 0 | (1-7M)/2 | M | (9M-1)/2 | 0 |  |

Since Min {Cj -Zj} = (1-7M)/2 therefore the corresponding column s1 enter into the basis and from the minimum ratio column, the Min {min. ratio} = 0 therefore the corresponding row A2 leave from the basis.

Table 3:

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Basic  Variables  (BV) | Cost of BV | Cj | 1 | 1 | 0 | 0 | M | M | Min. Ratio |
| Value of BV (Xb) | x1 | x2 | s1 | s2 | A1 | A2 |
| x2 | 1 | 1 | 1/7 | 1 | 0 | -1/7 | 0 | 1/7 |  |
| s1 | 0 | 0 | -5/7 | 0 | 1 | -2/7 | -1 | 2/7 |  |
| Z=1 | | Zj | 1/7 | 1 | 0 | -1/7 | 0 | 1/7 |  |
| Cj - Zj | 6/7 | 0 | 0 | 1/7 | M | M-(1/7) |  |

Since all Cj – Zj ≥ 0, therefore the iteration process end. The optimum solution LPP is x1 = 0,

x2 = 1 and the Min z = 1

**Problems to be discussed by the faculty:**

1. Solve the following problem using the Big M of the simplex method:

Maximize *z* = 2*x*1 + *x*2 + *x*3,

subject to:

2*x*1 + 3*x*2 − *x*3 ≤ 9,

2*x*2 + *x*3 ≥ 4,

*x*1 + *x*3 = 6,

*x*1, *x*2, *x*3 ≥ 0.

2. Apply the phase I simplex method to find a feasible solution to the problem:

*x*1 − 2*x*2 + *x*3 = 2,

−*x*1 − 3*x*2 + *x*3 = 1,

2*x*1 − 3*x*2 + 4*x*3 = 7,

*x*1, *x*2, *x*3 ≥ 0.

**Practice problems**

1. Obtain the solution of the following Linear programming problem using Big-M method

Min Z=4x1+2x2,

subject to the constraints:

3x1+x2 ≥27,

x1+x2≥21,

x1, x2 ≥ 0.

2. Determine the solution of the following linear programming problem using Big M method

Max Z=3x1-x2

subject to the constraints:

2x1+x2 ≥ 2,

x1+3x2 ≤ 2, and

x2 ≤ 4, and

x1, x2 ≥ 0.

3. Determine the solution of the following linear programming problem using Big M method

Minimise *Z* = 8*x* + 10*y*

subject to the constraints:

3*x* + 9*y* ≥100

8*x* + 4*y* ≥150

And both *x* and *y* ≥0

4. Solve the following LPP using Big M Method.

Minimize z = 4x1 + 3x2

subject to the constraints:

            2x1+ x2 ≥ 10,

-3x1, + 2x2 ≤ 6

             x1 + x2 ≥ 6,

x1 ≥ 0 and x2 ≥ 0.

5. Solve the following LPP using Big M Method.

Maximize z = 3x1, + 2x2

Subject to the constraints:

            2x1 + x2 ≤ 2,

3x1 + 4x2 ≥ 12,

x1, x2 ≥ 0.

6. Solve the following LPP using Big M Method.

Minimize Z = 3x1 + x2  
 Subject to constraints  
 4x1 + x2 = 4   
 5x1 + 3x2 ≥ 7   
 3x1 + 2x2≤ 6   
 where x1, x2≥0